Paraxial Lens Formulas

PARAXIAL FORMULAS FOR LENSES IN AIR

The following formulas are based on the behavior of paraxial rays, which are always very close and nearly parallel to the optical axis. In this region, lens surfaces are always very nearly normal to the optical axis, and hence all angles of incidence and refraction are small. As a result, the sines of the angles of incidence and refraction are small (as used in Snell's law) and can be approximated by the angles themselves (measured in radians).

The paraxial formulas do not include effects of spherical aberration experienced by a marginal ray — a ray passing through the lens near its edge or margin. All EFL values (f) tabulated in this catalog are paraxial values which correspond to the paraxial formulas. The following paraxial formulas are valid for both thick and thin lenses unless otherwise noted. The refractive index of the lens glass, n, is the ratio of the speed of light in vacuum to the speed of light in the lens glass. All other variables are defined in figure 1.33.

Focal Length

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{(n-1)^2}{n}\,\frac{t_c}{r_1 r_2} \qquad (1.33)$$

where n is the refractive index, t_c is the center thickness, and the sign convention previously given for the radii r_1 and r_2 applies. For thin lenses, $t_c \cong 0$, and for plano lenses either r_1 or r_2 is infinite. In either case the second term of the above equation vanishes, and we are left with the familiar lens maker's formula:

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad (1.34)$$

Surface Sagitta and Radius of Curvature

(refer to figure 1.34)

$$r^2 = (r-s)^2 + \left(\frac{d}{2}\right)^2 \qquad (1.35)$$

$$s = r - \sqrt{r^2 - \left(\frac{d}{2}\right)^2} > 0 \qquad (1.36)$$

$$r = \frac{s}{2} + \frac{d^2}{8s}. \qquad (1.37)$$

An often useful approximation is to neglect $s/2$.

Symmetric Lens Radii ($r_2 = -r_1$)

With center thickness constrained,

$$r_1 = (n-1)\left[f \pm \sqrt{f^2 - \left(\frac{f t_c}{n}\right)}\right] \qquad (1.38)$$

$$= (n-1)\,f\left[1 + \sqrt{1 - \left(\frac{t_c}{nf}\right)}\right]$$

where, in the first form, the + sign is chosen for the square root if f is positive, but the − sign must be used if f is negative. In the second form, the + sign must be used regardless of the sign of f.

Plano Lens Radius

Since r_2 is infinite,

$$r_1 = (n-1)\,f. \qquad (1.39)$$

Principal-Point Locations (signed distances from vertices)

$$A_2H'' = \frac{-r_2 t_c}{n(r_2 - r_1) + t_c(n-1)} \qquad (1.40)$$

$$A_1H = \frac{-r_1 t_c}{n(r_2 - r_1) + t_c(n-1)} \qquad (1.41)$$

where the above sign convention applies.

For symmetric lenses ($r_2 = -r_1$),

$$A_1H = -A_2H''$$

$$= \frac{r_1 t_c}{2nr_1 - t_c(n-1)}. \qquad (1.42)$$

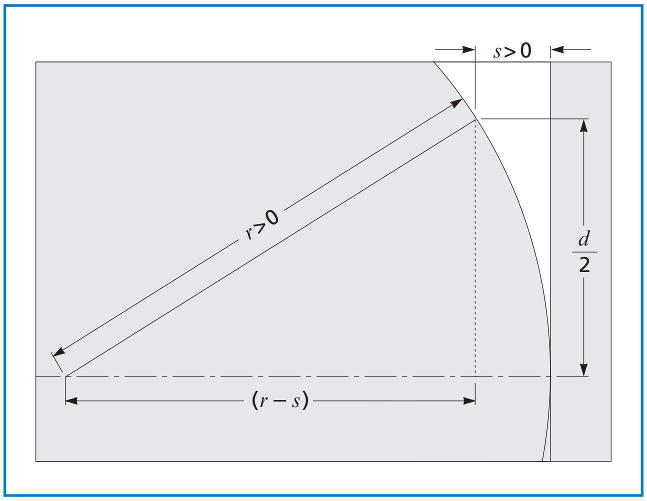

Figure 1.34 **Surface sagitta and radius of curvature**

If either r_1 or r_2 is infinite, l'Hôpital's rule from calculus must be used. Thus, referring to *Aberration Balancing*, for plano-convex lenses in the correct orientation,

$$A_1H = 0$$

and (1.43)

$$A_2H'' = -\frac{t_c}{n}.$$

For flat plates, by letting $r_1 \rightarrow \infty$ in a symmetric lens, we obtain $A_1H = A_2H'' = t_c/2_n$. These results are useful in connection with the following paraxial lens combination formulas.

Hiatus or Interstitium (principal-point separation)

$$HH'' = t_c\left\{1 - \frac{f}{n}\left[\frac{1}{f} - \frac{(n-1)^2}{n}\frac{t_c}{r_1 r_2}\right]\right\} \qquad (1.44)$$

which, in the thin-lens approximation (exact for plano lenses), becomes

$$HH'' = t_c\left(1 - \frac{1}{n}\right). \qquad (1.45)$$

Solid Angle

The solid angle subtended by a lens, for an observer situated at an on-axis image point, is

$$\Omega = 2\pi(1-\cos\theta) = 4\pi\sin^2\left(\frac{\theta}{2}\right) \qquad (1.46)$$

where this result is in steradians, and where

$$\theta = \arctan\left(\frac{CA}{2s}\right) \qquad (1.47)$$

is the apparent angular radius of the lens clear aperture. For an observer at an on-axis object point, use s instead of s''. To convert from steradians to the more intuitive sphere units, simply divide Ω by 4π. If the Abbé sine condition is known to apply, θ may be calculated using the arc sine function instead of the arctangent.

Back Focal Length

$$f_b = f'' + A_2H'' = f'' - \frac{r_2 t_c}{n(r_2 - r_1) + t_c(n-1)} \qquad (1.48)$$

where the sign convention presented above applies to A_2H'' and to the radii. If r_2 is infinite, l'Hôpital's rule from calculus must be used, whereby

$$f_b = f'' - \frac{t_c}{n}. \qquad (1.49)$$

Front Focal Length

$$f_f = f - A_1H = f + \frac{r_1 t_c}{n(r_2 - r_1) + t_c(n-1)} \qquad (1.50)$$

where the sign convention presented above applies to A_1H and to the radii. If r_1 is infinite, l'Hôpital's rule from calculus must be used, whereby

$$f_f = f - \frac{t_c}{n}. \qquad (1.51)$$

Edge-to-Focus Distances

For positive lenses,

$$A = f_f + s_1 \qquad (1.52)$$

and

$$B = f_b + s_2 \qquad (1.53)$$

where s_1 and s_2 are the sagittas of the first and second surfaces. Bevel is neglected.

Magnification or Conjugate Ratio

$$m = \frac{s''}{s} = \frac{f}{s-f} = \frac{s''-f}{f}. \qquad (1.54)$$

PARAXIAL FORMULAS FOR LENSES IN ARBITRARY MEDIA

These formulas allow for the possibility of distinct and completely arbitrary refractive indices for the object space medium (refractive index n'), lens (refractive index n''), and image space medium (refractive index n). In this situation, the EFL assumes two distinct values, namely f in object space and f'' in image space. It is also necessary to distinguish the principal points from the nodal points. The lens serves both as a lens and as a window separating the object space and image space media.

The situation of a lens immersed in a homogenous fluid (figure 1.35) is included as a special case ($n = n''$). This case is of considerable practical importance. The two values f and f'' are again equal, so that the lens combination formulas are applicable to systems immersed in a common fluid. The general case (two different fluids) is more difficult, and it must be approached by ray tracing on a surface-by-surface basis.

Lens Constant (k)

This number appears frequently in the following formulas. It is an explicit function of the complete lens prescription (both radii, t_c and n') and both media indices (n and n''). This dependence is implicit anywhere that k appears.

$$k = \frac{n'-n}{r_1} + \frac{n''-n'}{r_2} - \frac{t_c(n'-n)(n''-n')}{n' r_1 r_2}. \quad (1.55)$$

Effective Focal Lengths

$$f = \frac{n}{k} \qquad f'' = \frac{n''}{k}. \quad (1.56)$$

Lens Formula (Gaussian form)

$$\frac{n}{s} + \frac{n''}{s''} = k. \quad (1.57)$$

Lens Formula (Newtonian form)

$$xx'' = f''f = \frac{nn''}{k^2}. \quad (1.58)$$

Principal-Point Locations

$$A_1H = \frac{nt_c}{k}\left(\frac{n''-n'}{n'r_2}\right) \quad (1.59)$$

$$A_2H'' = \frac{-n''t_c}{k}\left(\frac{n'-n}{n'r_1}\right). \quad (1.60)$$

Object-to-First-Principal-Point Distance

$$s = \frac{ns''}{ks''-n''}. \quad (1.61)$$

Second Principal-Point-to-Image Distance

$$s'' = \frac{n''s}{ks-n}. \quad (1.62)$$

Magnification

$$m = \frac{ns''}{n''s}. \quad (1.63)$$

Lens Maker's Formula

$$\frac{n}{f} = \frac{n''}{f''} = k. \quad (1.64)$$

Nodal-Point Locations

$$A_1N = A_1H + HN \quad (1.65)$$

$$A_2N'' = A_2H'' + H''N''. \quad (1.66)$$

Separation of Nodal Point from Corresponding Principal Point

$HN = H''N'' = (n''-n)/k$, positive for N to right of H and N″ to right of H″.

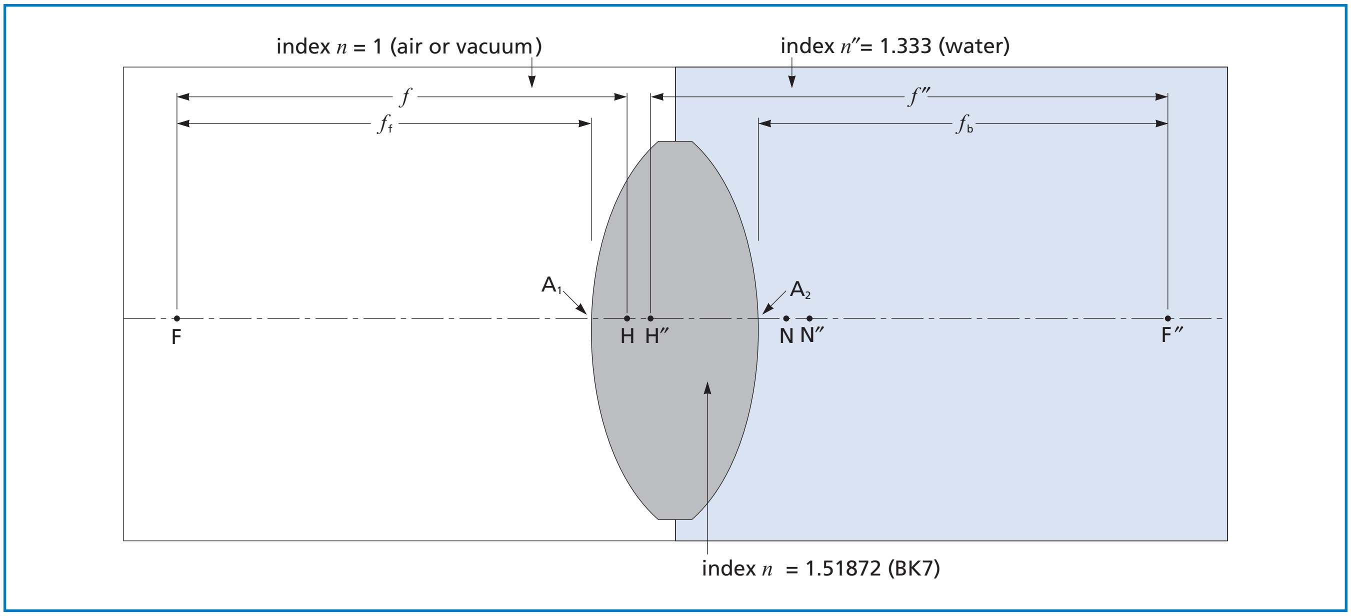

Figure 1.35 **Symmetric lens with disparate object and image space indexes**

Back Focal Length

$$f_b = f'' + A_2H''. \quad \text{(see eq. 1.48)}$$

Front Focal Length

$$f_f = f - A_1H. \quad \text{(see eq. 1.50)}$$

Focal Ratios

The focal ratios are f/CA and f''/CA, where CA is the diameter of the clear aperture of the lens.

Numerical Apertures

$$n \sin\theta$$

$$\text{where } \theta = \arcsin\left(\frac{CA}{2s}\right)$$

and

$$n'' \sin\theta''$$

$$\text{where } \theta'' = \arcsin\left(\frac{CA}{2s''}\right).$$

Solid Angles (in steradians)

$$\Omega = 2\pi\,(1-\cos\theta) \quad \text{(see eq. 1.46)}$$

$$= 4\pi \sin^2\left(\frac{\theta}{2}\right)$$

$$\text{where } \theta = \arctan\left(\frac{CA}{2s}\right)$$

$$\Omega = 2\pi(1 - \cos\theta'')$$

$$= 4\pi \sin^2\left(\frac{\theta}{2}\right)$$

$$\text{where } \theta'' = \arctan\left(\frac{CA}{2s''}\right).$$

To convert from steradians to spheres, simply divide by 4π.

APPLICATION NOTE

For Quick Approximations

Much time and effort can be saved by ignoring the differences among f, f_b, and f_f in these formulas by assuming that $f = f_b = f_f$. Then s becomes the lens-to-object distance; s'' becomes the lens-to-image distance; and the sum of conjugate distances $s + s''$ becomes the object-to-image distance. This is known as the thin-lens approximation.

APPLICATION NOTE

Physical Significance of the Nodal Points

A ray directed at the primary nodal point N of a lens appears to emerge from the secondary nodal point N″ without change of direction. Conversely, a ray directed at N″ appears to emerge from N without change of direction. At the infinite conjugate ratio, if a lens is rotated about a rotational axis orthogonal to the optical axis at the secondary nodal point (i.e., if N″ is the center of rotation), the image remains stationary during the rotation. This fact is the basis for the nodal slide method for measuring nodal-point location and the EFL of a lens. The nodal points coincide with their corresponding principal points when the image space and object space refractive indices are equal ($n = n''$). This makes the nodal slide method the most precise method of principal-point location.